Theorems of Derivatives Worksheet - Answer Key

1. For $f(x) = e^x$, find a point in $[-3, 3]$ where $f'(x) = 0$.
   It does not exist because the condition (3) of Rolle's theorem is not satisfied ($f(-3) \neq f(3)$).

2. For $f(x) = x^2 - 4x + 5$, find the minimum value of $f(x)$ for x in $[-1, 5]$.
   $f(-1) = f(5) = 10$, by Rolle's theorem the upward facing parabola $f(x)$ will achieve a minimum at $a = 2$ since $f'(2) = 0$. The minimum value occurs at $\mathbf{f(2) = 1}$.

3. Are all the conditions for Rolle's theorem satisfied for $f(x) = \tan(x)$ in the interval $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$?
   $\tan(x)$ is discontinuous and not differentiable at $x = \frac{\pi}{2}$.

4. For $f(x) = 4x^3 + 21x^2 - 60x + 3$, can you use Rolle's theorem to find a value of x such that $f'(x) = 0$.
   No, $f(x)$ is increasing so we cannot find an interval $[a, b]$ such that $f(a) = f(b)$.

5. Find an interval where $f(x) = \sin(x) + 2x$ assumes the value 1.
   $\left(0, \frac{\pi}{2}\right)$, using the intermediate value theorem: $f(0) < 1$ and $f\left(\frac{\pi}{2}\right) > 1$.

6. Find an interval where Mean Value Theorem can be applied for $f(x) = \frac{1}{x^3 - 2x^2 + x}$.
   The Mean Value Theorem can be applied for any interval not containing 0 and 1.

7. Find a value of c such that the conclusion of Mean Value Theorem is satisfied for $f(x) = 2x^2 - 3x + 1$ on $[-1, 1]$.
   $f'(c) = \frac{f(1) - f(-1)}{1 - (-1)} = -3 \implies 4c - 3 = -3 \implies \mathbf{c = 0}$

8. Find the upper bound on an increasing fn $f(a)$ if $f(0) = 0$ and $f'(x) \leq 1$ on $[0, a]$.
   $f(a) - f(0) = (a - 0)f'(c) < a(1) \implies \mathbf{f(a) < a}$

9. Prove that $\cos(x) < 2x$ for $x \geq 1$.
   $h(x) = 2x - \cos(x)$, $h(1) > 0$ and $h'(x) = 1 + \sin(x) > 0 \implies h(x) = h(1) + (x - 1)h'(x) > 0$

10. Nate starts driving 10 miles from Shmoop at a speed lying between 40 and 50mph. Find a bound on his distance from Shmoop 3hrs later.
    $3(40) \leq d(3) - d(0) = 3d'(t) \leq 3(50) \implies \mathbf{130 \leq d(3) \leq 160}$
